Problem Set
Linear Algebra

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1. Show that the sets $V_1 = \{0_{\mathbb{R}^n}\}$ and $V_2 = \mathbb{R}^n$ are vector subspace of $\mathbb{R}^n$.
2. Let $V = \{X \in \mathbb{R}^n | X_n = 0\}$, show that V is a vector subspace of $\mathbb{R}^n$.
3. Let V_1 and V_2 be both vector subspace of $\mathbb{R}^n$, show that $V_1 \cap V_2$ is also a vector subspace of $\mathbb{R}^n$.

Solution: If V_1 and V_2 are v.s. then for any $\hat{v}_1, \tilde{v}_1 \in V_1$, $\hat{v}_1 + \lambda\tilde{v}_1 \in V_1$, and for any $\hat{v}_2, \tilde{v}_2 \in V_2$, $\hat{v}_2 + \lambda\tilde{v}_2 \in V_2$, for any $\lambda \in \mathbb{R}$. Take any $\hat{v}, \tilde{v} \in V_1 \cap V_2$, and $\lambda \in \mathbb{R}$, then $v = \hat{v} + \lambda\tilde{v}$. If $\hat{v}, \tilde{v} \in V_1 \cap V_2$ then $\hat{v}, \tilde{v} \in V_1$, so $v \in V_1$ because V_1 is v.s. Do the same for V_2. Then, $v \in V_1 \cap V_2$, so $V_1 \cap V_2$ is a v.s.

4. Let $X_0 \in \mathbb{R}^n$, and let $V_{X_0} = \{\alpha X_0 | \alpha \in \mathbb{R}\}$. Show that V_{X_0} is a vector subspace $\mathbb{R}^n$.
5. Let $X_0, X_1 \in \mathbb{R}^n$. Show that $V_{X_0} \cap V_{X_1} \neq \{0_{\mathbb{R}^n}\}$ if and only if there is a scalar $\lambda \neq 0$ such that $X_0 = \lambda X_1$.
6. Let $X_1^t = (1, 2, 3)$ and $X_2^t = (4, 5, 6)$, check if $X = (10, 11, 12)$ is an element of $L\{X_1, X_2\}$.

Solution: If $X \in L\{X_1, X_2\}$, there there are α and β in $\mathbb{R}$ such that

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix}$$

This leaves the system:

$$\begin{aligned} a + 4\beta &= 10 \\ 2\alpha + 5\beta &= 11 \\ 3\alpha + 6\beta &= 12 \end{aligned}$$

Replacing the first one in the other equations:

$$\begin{aligned} \alpha + \beta &= 1 \\ 2\alpha + 2\beta &= 2 \end{aligned}$$

So basically, for any α, β such that $\alpha + \beta = 1$ the condition is satisfied. Replace back in the first equation. $\alpha + 4\beta = 10$, we get now $1 + 3\beta = 10$ or $\beta = 3$, and therefore $\alpha = -2$.

7. Show that two vector sub space V_1, V_2 of $\mathbb{R}^n$ it holds that $dim(V_1 \cap V_2) \leq \min\{dim(V_1), dim(V_2)\}$

8. Let the vector subspace $V_1 = \{X \in \mathbb{R}^n | X_n = 0\}$ of $\mathbb{R}^n$. Find $dim(V_1)$. Analogously do the same for the vector subspace $V_2 = \{X \in \mathbb{R}^n | \sum_{i=1}^{n} a_i X_i = 0\}$, given some α_is.

9. Show that for any $X \in \mathbb{R}^n$, $0 \perp X$. Show also that if $X \in \mathbb{R}^n$ is such that $Z \perp X$, for any $X \in \mathbb{R}^n$, then $Z = 0$.

> **Solution:**
> First part:
>
> $$0 \perp X \;\Leftrightarrow\; 0 \cdot X = 0$$
> $$0 \cdot X = \sum 0 \times x_i = 0$$
>
> Second part:
>
> $$Z \cdot X = \sum z_i \times x_i = 0$$
>
> If this is true for any $\mathbb{R}^n$, then in particular is true for the vectors of the canonical basis of $\mathbb{R}^n$. Let C^i represent each of these vectors, with 1 in the ith component, and 0 everywhere else. The inner product would be:
>
> $$Z \cdot C_i = z_i \times c_i = z_i \times 1 = z_i$$
>
> Doing the same for every vector in the canonical basis, we get that $z_i = 0$ for every i, and therefore $Z = 0$.

10. Show that if $X \perp X_i$, with $i = 1, 2, ..., k$, then $X \perp Y$, for any $Y \in L(\{X_1, X_2, ..., X_k\})$.

11. Show that $\hat{X} = \frac{X}{||X||}$ is a unit vector, for $X \neq 0$.

12. Consider the family of vectors $\mathcal{B} = \{e_1, e_2, ..., e_n\} \subseteq \mathbb{R}^n$ where $e_i = (0, ..., 0, 1, 0, ..., 0)$. Show that $\mathcal{B}$ is a family of unit vector that are mutually perpendicular. Show also that for any $X \in \mathbb{R}^n$, it holds that $\sum_{j=1}^{n} (X \cdot e_j) e_j$.

13. Show that if X, Y are two perpendicular vectors, different from zero, then X and Y are linearly independent.

> **Solution:** Let X and Y be linearly dependent, then $\exists\, a, b \neq 0$ such that $aX + bY = 0$ or $X = \frac{-b}{a} Y$.
>
> $$X \cdot Y = \sum x_i \times y_i = \sum \frac{-b}{a} y_i \times y_i = \frac{-b}{a} \sum y_i^2$$
>
> But, as $Y \neq 0$, then at least for one i, $y_i^2 \neq 0$, and therefore $X \cdot Y \neq 0$, so the vectors are not perpendicular.

14. Let $X_1, X_2, ..., X_n$ be non zero vectors and orthogonal among them, show that $\{X_1, X_2, ..., X_N\}$ as a basis of $\mathbb{R}^n$. If further, we assume that these vectors are unit vectors, show then that for any $X \in \mathbb{R}^n$ it holds that $X = \sum_{j=1}^{n} (X \cdot X_j) x_j$.

15. Show that the matrices, with the sum and scalar multiplication, is a vector subspace.

16. Given the matrix

$$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Show that A is invertible if and only if $\alpha \neq 0$.

> **Solution:**
>
> $$\det(A) = 4\alpha$$
>
> (A invertible $\Rightarrow \alpha \neq 0$) We know that A is invertible if and only if $\det(A) \neq 0$, so given that $\det(A) = 4\alpha$, we know that $4\alpha \neq 0$, and therefore $\alpha \neq 0$.
>
> ($\alpha \neq 0 \Rightarrow A$ invertible) Again, this is equivalent to A not invertible $\Rightarrow \alpha = 0$. We know that A not invertible if and only if $\det(A) = 0$, therefore $4\alpha = 0$, or $\alpha = 0$.

17. Show that, for a given matrix A, the rank of A is the same that the rank of its transpose.

18. Let A be an upper triangular matrix. Show that its rank coincides with the number of non zero elements that lie on its diagonal.

19. Let $Y^t = (1, 2, 3, 4)$, $X^t = (1, 1, 1, 1)$, $X_2^t = (0, 1, 0, 1)$ in $\mathbb{R}^4$. Find $proj_V(Y)$ with $V = L\{X_1, X_2\}$.

20. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & \beta & 4 \\ 0 & 3 & \alpha \end{pmatrix}$$

Find conditions over α and β such that A is invertible.

> **Solution:** $\det(A) = \beta\alpha + 0 + 18 - 0 - 12 - 4\alpha = \alpha\beta - 4\alpha + 6$. What is necessary is that $\det(A) \neq 0$, therefore $\alpha\beta - 4\alpha + 6 \neq 0$. If $\alpha\beta - 4\alpha + 6 = 0$, then $\alpha(4 - \beta) = 6$ so for A to be invertible, it is necessary that $\alpha \neq \frac{6}{4-\beta}$ or that $\beta = 4$.

21. Let $A \in \mathbb{R}^{n \times n}$, be such that its eigenvalues are different between them, and also different from zero. Let V the matrix composed with the eigenvectors, that is first column of V is the eigenvector associated to the first eigenvalue.
    (a) Show that $V^{-1}AV = D(\lambda)$, being $D(\lambda)$ the diagonal matrix whose elements are the eigenvalues of A.
    (b) Show that A is invertible if and only if all its eigenvalues are different from zero.
    (c) Show that for any $n \in \mathbb{N}$, it holds that $A^n = VD(\lambda^n)V^{-1}$, where $D(\lambda^n)$ is the diagonal matrix with the eigenvalues of A raised to the power of n.

22. Let $A \in \mathbb{R}^n$ a diagonal matrix with values λ_i, $i = 1, ..., n$. Show that $det(A) = \prod_{i=1}^{n} \lambda_i$.

23. Show that the determinant of an upper triangular matrix is equal to the product of the elements on its diagonal.

24. Consider the following matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

.

(a) Find its eigenvalues and eigenvectors.

Solution: Find $\det(A - \lambda I)$ to solve the characteristic equation:

$$
\begin{aligned}
(A-\lambda I) &= \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix} \\
\det(A-\lambda I) &= (1-\lambda)^2(2-\lambda) - 2(1-\lambda) \\
&= (1-\lambda)\left[(1-\lambda)(2-\lambda) - 2\right] \\
&= (1-\lambda)\left[2 - 3\lambda + \lambda^2 - 2\right] \\
&= (1-\lambda)\left[-3\lambda + \lambda^2\right] \\
&= (1-\lambda)\lambda\left[-3+\lambda\right]
\end{aligned}
$$

So for the determinant to be zero, $\lambda = 0$, or $\lambda = 1$, or $\lambda = 3$. Let's find the eigenvectors associated to those eigenvalues.
Start with $\lambda = 0$

$$
\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

We obtain:

$$
\begin{aligned}
x_1 - x_2 &= 0 \\
-x_1 + x_2 - x_3 &= 0 \\
-x_2 + x_3 &= 0
\end{aligned}
$$

From where we obtain $x_1 = x_2 = x_3$, so the vector $(1,1,1)^T$ is the eigenvector associated to $\lambda = 0$, in particular for $k = 1$. Now lets look for the eigenvector associated to the eigenvalue $\lambda = 1$
We obtain:

$$
\begin{aligned}
x_1 - x_2 &= x_1 \\
-x_1 + x_2 - x_3 &= x_2 \\
-x_2 + x_3 &= x_3
\end{aligned}
$$

From where we get $x_2 = 0$ from the first equation, $x_1 = -x_3$. Then the vector $(1,0,-1)^T$ would be an eigenvector associated to $\lambda = 1$. Finally, when $\lambda = 3$ we obtain:

$$
\begin{aligned}
x_1 - x_2 &= 3x_1 \\
-x_1 + x_2 - x_3 &= 3x_2 \\
-x_2 + x_3 &= 3x_3
\end{aligned}
$$

Which leads $x_2 = -2x_1 = -2x_3$, so $x_1 = x_3$. Then the vector $(1,-2,1)^T$ would be the final eigen vector associated to the eigenvalue $\lambda = 3$.

(b) Find A^5

Solution: Having all the eigenvalues and eigenvectors we can write the matrix decomposition:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

And given that $A = VDV^{-1}$.

$$\begin{aligned} A^5 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1^5 & 0 \\ 0 & 0 & 3^5 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 243 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \\ &= \begin{pmatrix} 41 & -81 & 40 \\ -81 & 162 & -81 \\ 40 & -81 & 41 \end{pmatrix} \end{aligned}$$

25. Let A be a positive semidifinite matrix. Show that there is a matrix R such that A can be written as $A = R^t R$.

Solution: As A is positive semidefinite, then we know there is a diagonal matrix with its eigenvalues, all positive, and an orthogonal matrix V such that $A = VDV^t$. Define $H = D(\sqrt{\lambda})$, then $A = VHHV^t$, but as H is also diagonal, $H = H^t$, so $A = VHH^tV^t = VH(VH)^t$. Let $R = VH$.

26. Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $f(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m}$.

27. Show that if $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are linear functions, then $f + g$ is also linear.

Solution: f and g linear, then for any $X, Y \in \mathbb{R}^n$ $f(X+Y) = f(X)+f(Y)$, $g(X+Y) = g(X)+g(Y)$, $f(\alpha X) = \alpha f(X)$, and $g(\alpha X) = \alpha g(X)$ for $\alpha \in \mathbb{R}$.

Then, $[f+g](X+Y) = f(X+Y)+g(X+Y) = f(X)+f(Y)+g(X)+g(Y) = [f+g](X)+[f+g](Y)$.

Also, $[f + g](\alpha X) = f(\alpha X) + g(\alpha X) = \alpha f(X) + \alpha g(X) = \alpha(f(X) + g(X)) = \alpha[f + g](X)$

Concluding, $f + g$ is linear.

28. Show that if $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are linear functions, then $f \circ g$ is also linear. Indeed, show that $[f \circ g] = [f][g]$.